En passant à la limite, on a donc :

$$(-1+\exp(ia\pi))\int_{-\infty}^{\infty}\frac{\exp(ax)}{ch(x)}\,dx = 2\pi i\exp\left(i\frac{a\pi}{2}\right)$$

$$\Rightarrow \exp\left(i\frac{a\pi}{2}\right)\left(-\exp\left(-\frac{ia\pi}{2}\right)+\exp\left(\frac{ia\pi}{2}\right)\right)\int_{-\infty}^{\infty}\frac{\exp(ax)}{ch(x)}\,dx = 2\pi i\exp\left(i\frac{a\pi}{2}\right)$$

$$\sin\left(\frac{a\pi}{2}\right)\int_{-\infty}^{\infty}\frac{\exp(ax)}{ch(x)} = \pi \Rightarrow \int_{-\infty}^{\infty}\frac{\exp(ax)}{ch(x)}\,dx = \frac{\pi}{\sin\left(\frac{a\pi}{2}\right)}$$

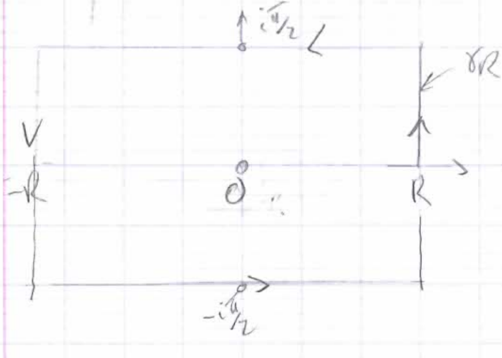
Ex 6.4.7

$$I_{a,b} = \int_{-\infty}^{\infty}\frac{\exp(ax)-\exp(bx)}{sh(x)}\,dx \underset{conv}{\Longleftrightarrow} |a|<1,\ |b|<1$$

On pose $f_{a,b}(z) = \dfrac{\exp(az)-\exp(bz)}{sh(z)}$ $\forall z\in\mathbb{C}\setminus Z$

Avec $Z = \{ik\pi,\ k\in\mathbb{Z}\}$

On part de $J_{a,b}(R) = \int_{\gamma_R} f_{a,b}(z)\,dz$



$$H_1(R) = \int_{-R}^{R} f_{a,b}\left(x+i\frac{\pi}{2}\right)dx$$

$$H_2(R) = \int_{-\pi/2}^{\pi/2}\frac{\exp(a(R+iy))-\exp(b(R+iy))}{sh(R+iy)}\,dy$$

$$H_3(R) = \int_{R}^{-R} f_{a,b}\left(x-i\frac{\pi}{2}\right)dx$$

$$H_4(R) = \int_{\pi/2}^{-\pi/2} f_{a,b}(-R+iy)\,dy$$

$$|H_2(R)| \leq \int_{-\pi/2}^{\pi/2}\frac{\exp(aR)+\exp(bR)}{sh(R)}\,dy = \pi\,\frac{\exp(aR)+\exp(bR)}{sh(R)} \xrightarrow[R\to\infty]{} 0$$

$$|H_3(R)| \leq \int_{-\pi/2}^{\pi/2}\frac{\exp(-aR)+\exp(-bR)}{sh(R)}\,dy = \pi\,\frac{\exp(-aR)+\exp(-bR)}{sh(R)} \xrightarrow[R\to\infty]{} 0$$

On remarque que : $sh\left(x+\frac{i\pi}{2}\right) = \dfrac{ie^{x}+ie^{-x}}{2} = i\,ch(x)$

$sh\left(x-\frac{i\pi}{2}\right) = \dfrac{-ie^{x}-ie^{-x}}{2} = -i\,ch(x)$